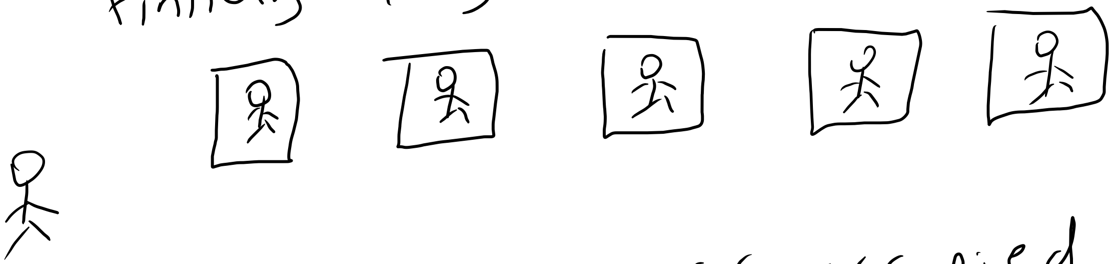
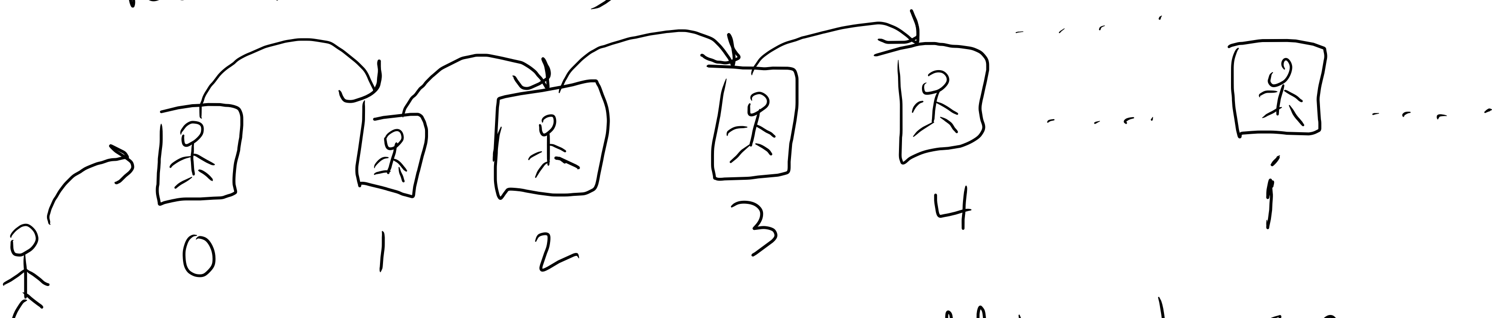
Infinity

- Lets say you have a hotel with finitely many rooms, e.g. 5 rooms.



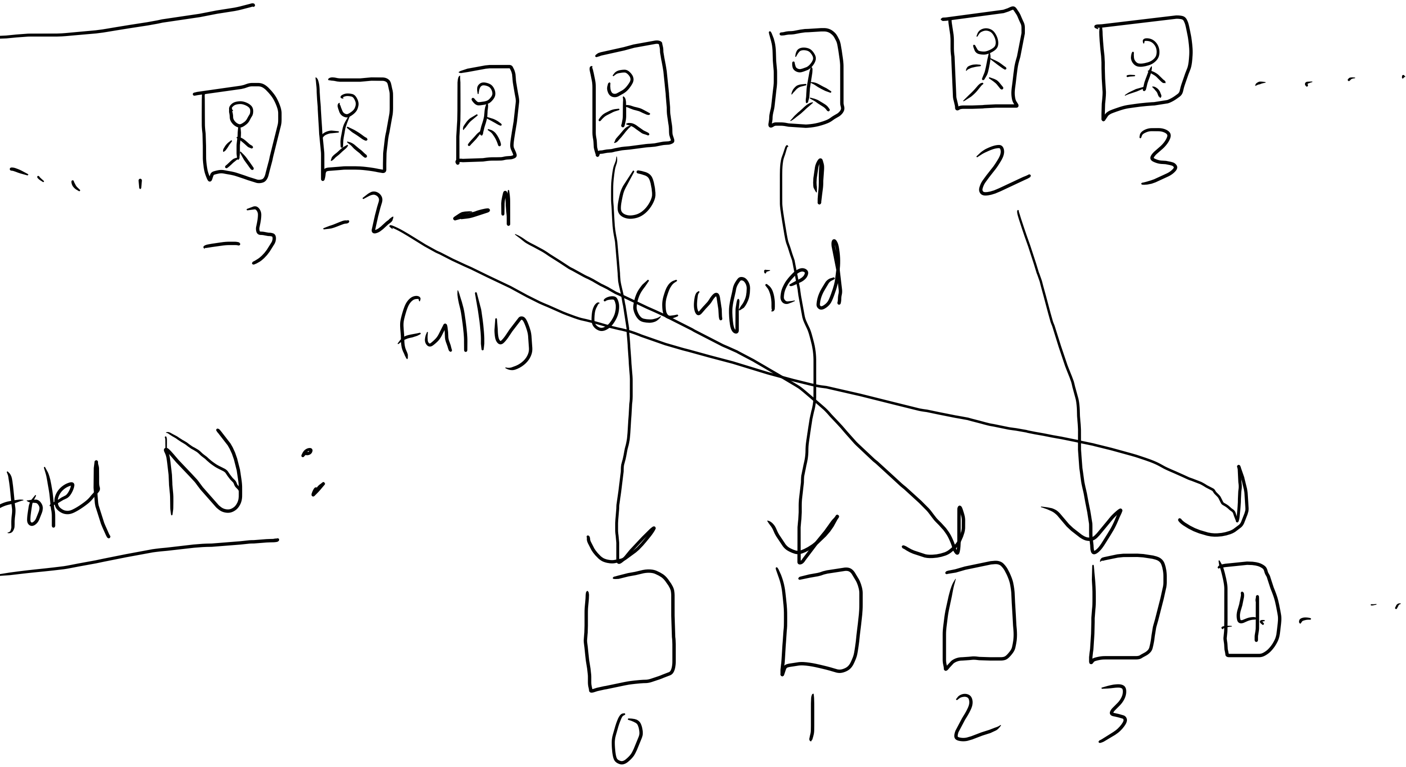
- Suppose all rooms are occupied
- Suppose one more person shows up, demands a room.
- Can't accomodate! $6 > 5$

- Infinite sets behave differently
- Suppose now the hotel has infinitely many rooms, indexed by $\mathbb{N}$.



- Suppose all occupied, but additional person shows up demanding a room.
- Put him in room 0, everyone else moves one room to the right: $i \to i+1$.

Hotel $\mathbb{Z}$:



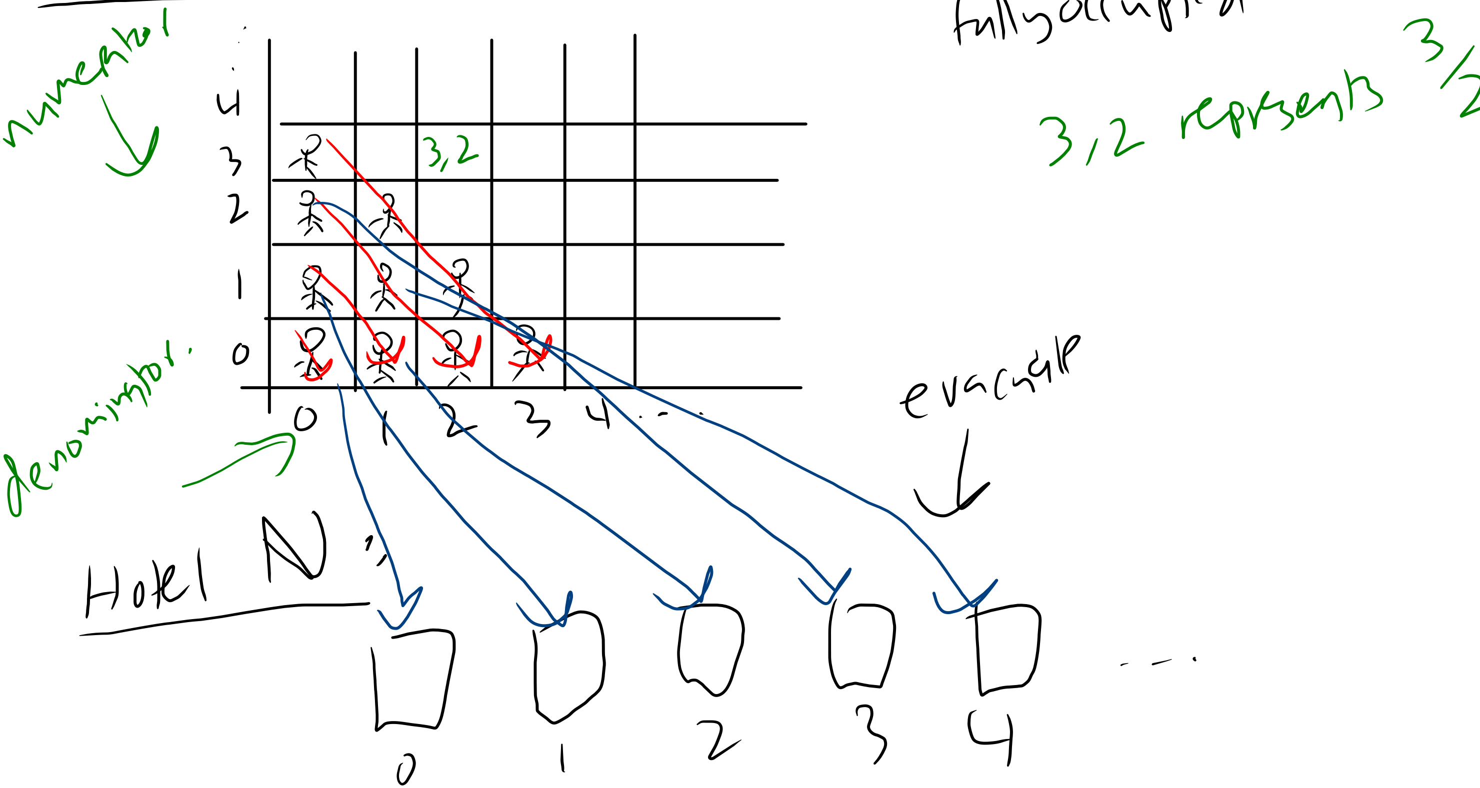
Can I evacuate $\mathbb{Z}$ to $\mathbb{N}$?

$\mathbb{N}$	$\mathbb{Z}$
0	0
1	1
2	-1
3	2
4	-2
5	3
6	-3
$\vdots$	

this is a one-to-one (a.k.a. bijection) correspondence between $\mathbb{Z}$ and $\mathbb{N}$.

So they are the same infinity.

Hotel $\mathbb{N} \times \mathbb{N}$:



Adding one, doubling, squaring doesn't change ∞.

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}|$$

- All of these are the "same" infinity.
- We call these sets <u>countably infinite</u>

More generally:

- Two sets have the same cardinality iff there is a bijection between them

$$|A| = |B| \text{ iff } \exists \text{ bijection } f: A \to B$$

- $|A| = k$ ($k \in \mathbb{N}$) iff $|A| = |\{1, 2, \ldots, k\}|$

  iff $\exists$ bijection $f: A \to \{1, \ldots, k\}$

- A is <u>finite</u> if $\exists k \in \mathbb{N}$ s.t. $|A| = k$
- A is infinite if it is not finite.

- we saw several infinite sets, but they all had same cardinality as $\mathbb{N}$.
- we call these countably infinite sets.
- <u>Countable set</u>: finite or countably infinite.

So far all infinite sets we saw were the same kind of infinity. I.e. they were all countably infinite.

Q: Is there a bigger infinity?
I.e. Is there a set which is not countable.

A: $\mathbb{R}$ is not countable.

we will show this.

$P(\mathbb{N})$

⋮

Thm: $\mathbb{R}$ are not countable.

Proof:

Represent $x \in \mathbb{R}$ in binary

$2\frac{1}{4} \to 10.01$

$\frac{1}{16} \to 0.0001$

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$

$\frac{1}{3} \to 0.0101010101\ldots$

$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

- Want to list all $\mathbb{R}$ in correspondence with $\mathbb{N}$.
- Make this even easier: Count real numbers >0 and <1

$$0.1011011\ldots$$

- Assume for contradiction this is possible.

0	1	0	1	1	0	1	...			
1	0	1	0	1	0	1	...			
2	1	0	0	1	0	0	1	0	0	
3	1	0	1	1	1	0	1	1	1	...
4	1	1	0	0	0	1	0	0	0	1 ...
5						1				
6										
⋮										

Diagonal

0.110101...

Diagonal flipped

0.001010...

- The flipped diagonal disagrees with each row k on kth bit
- therefore, flipped diagonal represents a real number nowhere in this list.
- Contradiction. $\square$

Book does same thing for $P(N)$

$$|P(N)| = |\mathbb{R}|$$

- We showed that $|\mathbb{R}|$ is a "bigger" infinity than $|\mathbb{N}|$.

Turns out: $|\text{Algorithms}| = |\mathbb{N}|$

$|\text{Problems}| = |\mathbb{R}| = |P(\mathbb{N})|$

$$|\text{Problems}| > |\text{Algorithms}|$$

Halting problem is one specific example of a problem without Algorithm to solve it.